ON THE STABILITY OF PLANE PARALLEL FLOWS OF AN INHOMOGENEOUS FLUID

(OB USTOICHIVOSTI PLOSKOPARALLEL'NYKH POTOKOV NEODNORODNOI ZHIDKOSTI)

PMM Vol.24, No.2, 1960, pp. 249-257

L. A. DIKII
(Moscow)

(Received 25 September 1959)

The theory of the stability of a fluid with density stratification is worked out to a considerably lesser degree than the problem relating to the stability of a homogeneous incompressible fluid. Specific difficulties which arise in this theory are easily understood after recalling the situation which occurs in the study of the stability of plane parallel flows of a homogeneous fluid. If, as is usually done, solutions are sought in the form of traveling waves

$$\varphi(z)\, e^{i\alpha(x-ct)} \tag{0.1}$$

(here z is the coordinate in the direction transverse to the basic flow and x the coordinate in the longitudinal direction), then the Orr-Sommerfeld equation

$$(V-c)(\varphi''-\alpha^2\varphi)-V''\varphi=-\frac{i\nu}{\alpha}(\varphi^{\mathrm{IV}}-2\alpha^2\varphi''+\alpha^4\varphi) \tag{0.2}$$

is valid for the amplitude $\phi(z)$, where $V(z)$ is the undisturbed flow velocity. If viscosity is neglected, then

$$(V-c)(\varphi''-\alpha^2\varphi)-V''\varphi=0 \tag{0.3}$$

In spite of the fact that the "inviscid" equation is considerably simpler in form than the "viscous" one, even being of second order rather than fourth order, its use encounters the same difficulties in principle. To begin with, the boundary value problem which arises (α is fixed and c is an eigen-value) can either have no solutions at all or have insufficiently many of them, in the sense that the eigen-functions which correspond to the eigen-values do not form a complete system. In such a case, even though all of the eigen-functions are known, it is nevertheless impossible to answer the fundamental question of stability theory – how will an arbitrary disturbance which arises at a certain moment in the fluid develop: will it grow without limit or will it remain finite?

However, with the introduction of a nonvanishing viscosity (even if it is small) the situation is improved. The eigen-value problem then has a complete system of eigen-functions. (From a mathematical point of view, this is an eigen-value problem of a certain non-self-adjoint operator and the completeness of the eigen- and "adjoint" function can be established, for instance, with the help of a theorem of Keldysh [1, Theorem 1].)

However, it is possible to say even more, namely, if viscosity is not introduced, then even the very mathematical statement of the problem remains not quite definite. Namely, the "inviscid" equation for ϕ has a singular point at $V - c = 0$. In this connection its solutions are multivalues and the correct branch choice can be made only if solutions of the "inviscid" equation are considered as limiting solutions of the complete "viscous" equation (a discussion of this question is contained in Lin's book [2]).

Unfortunately, we shall not use such an approach in the case of a fluid with a density which varies with height. The Orr-Sommerfeld equation takes the following form:

$$(V-c)^2(\varphi''-\alpha^2\varphi)-(V-c)V''\varphi-\frac{g}{\rho_0}\frac{d\rho_0}{dz}\varphi+\frac{1}{\rho_0}\frac{d\rho_0}{dz}[(V-c)^2\varphi'-(V-c)V'\varphi]=$$
$$=-\frac{i\nu}{\alpha}[\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi+\frac{1}{\rho_0}\frac{d\rho_0}{dz}(\varphi''-\alpha^2\varphi)](V-c) \qquad (0.4)$$

It is easily seen that a singular point of the equation remains at $V - c = 0$ even in the presence of non-zero viscosity. Thus, the viscous equation at this point proves to be no better than the "inviscid" one, and the question of the selection of the branch of the solution is left open. Therefore, mechanical transference of a rule for the selection of the branch of a solution derived for a homogeneous fluid to the case of an inhomogeneous fluid, as Schlichting [3] does, appears to be groundless.

Thus, in the study of the stability of an inhomogeneous fluid the introduction of viscosity does not give tangible advantages, and if the Reynolds number is sufficiently large, it is better to set the viscosity at once equal to zero. Further, it is necessary to realize that the expansions of the solutions by the wave solutions (0.1) and the investigation of the characteristic frequencies are not ends in themselves but only tools for investigating the Cauchy problem of a partial differential equation. But if such wave solutions are not found, then it is necessary to solve the Cauchy problem for the development of arbitrary initial disturbances in some other way.

We will consider the motion to be stable if an arbitrary initial disturbance arising in a finite region of space remains bounded as time increases. Otherwise we will call the motion unstable. In the present paper it is successfully shown that such an investigation can be carried through, at least in special examples, even in cases for which there are no wave solutions.

The subject of [4] is similar to the present paper; in that paper a similar example has been analyzed; however, the authors have proceeded by a different path. Namely, they have not rejected the expansion of the solution by the wave form (0.1) in view of the simplicity which results from the complete separation of the variables. However, it is not expected that such a wave solution will be a solution of (0.4) in the strict sense of the word. It must only satisfy the equation everywhere except at the singular point where no junction conditions whatsoever are specified.

One can even show too many of such "almost eigen-" functions in the same sense in which there are "too many" eigen-values in the usual boundary problem if a boundary condition, corresponding in the present case to the junction condition at the singular point, is missing.

Thus, wave solutions are formally understood as certain easily-calculated functions by which one can try to expand the solution of the Cauchy problem. However, in [4] the basic fact of the completeness of the system of "almost eigen-" functions is left without proof. Therefore, the approach adopted in [4] still requires serious substantiation.

1. The following example will be considered: a two-dimensional horizontal flow in an unbounded half-space, whose velocity V increases linearly with height and whose density ρ_0 decreases exponentially:

$$V(z) = kz, \qquad \rho_0 = \mathrm{const}\, e^{-\beta z}$$

Taylor [5] studied this example by the wave method. He established that for values of the Richardson number $R = g\beta/k^2$ larger than 1/4 there exist neutral waves and that for smaller Richardson numbers waves do not in general exist.

From this many authors [6,7] have concluded that the flow is stable only for values of $R > 1/4$.

It is shown below that the solution is stable for all positive values of R (in the sense formulated earlier*).

---

* In the indicated paper [4] the same flow is examined, but between two solid walls. The conclusion with regard to the stability is the same.

Let ρ', p', u', w' be the perturbations of the density, the pressure and the velocity.

We shall write the linearized equations of motion

$$\begin{aligned}&\rho_0\left(\frac{\partial u'}{\partial t}+V\frac{\partial u'}{\partial x}+w'\frac{\partial V}{\partial z}\right)=-\frac{\partial p'}{dx}, &&\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0\\&\rho_0\left(\frac{\partial w'}{\partial t}+V\frac{\partial w'}{\partial x}\right)=-\frac{\partial p'}{\partial z}-g\rho', &&\frac{\partial \rho'}{\partial t}+V\frac{\partial \rho'}{\partial x}+w'\frac{d\rho_0}{dz}=0\end{aligned}\qquad(1.1)$$

If p' and ρ' are eliminated and the stream function ψ introduced, we then obtain

$$\beta\left(\frac{\partial}{\partial t}+kz\frac{\partial}{\partial x}\right)^2\frac{\partial\psi}{\partial z}-\beta k\left(\frac{\partial}{\partial t}+kz\frac{\partial}{\partial x}\right)\frac{\partial\psi}{\partial x}-\left(\frac{\partial}{\partial t}+kz\frac{\partial}{\partial x}\right)^2\Delta\psi=g\beta\frac{\partial^2\psi}{\partial x^2}$$
$$\left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2},\quad u'=-\frac{\partial\psi}{\partial z},\quad w'=\frac{\partial\psi}{\partial x}\right)$$

By the substitution of $e^{1/2\beta z}\chi$ we shall eliminate the term in $\partial\psi/\partial z$; we obtain

$$\left(\frac{\partial}{\partial t}+kz\frac{\partial}{\partial x}\right)^2\left(\frac{1}{4}\beta^2-\Delta\right)\chi-\beta k\left(\frac{\partial}{\partial t}+kz\frac{\partial}{\partial x}\right)\frac{\partial\chi}{\partial x}=g\beta\frac{\partial^2\chi}{\partial x^2}$$

We shall transform to a coordinate system which moves along with the mean flow: $t_1=t$, $z_1=z$, $x_1=x-kzt$. We will have

$$\frac{\partial^2}{\partial t_1^2}\left[\frac{1}{4}\beta^2-\frac{\partial^2}{\partial x_1^2}-\left(\frac{\partial}{\partial z_1}-kt_1\frac{\partial}{\partial x_1}\right)^2\right]\chi-\beta k\frac{\partial^2\chi}{\partial t_1\partial x_1}-g\beta\frac{\partial^2\chi}{\partial x_1^2}=0$$

In place of x_1, z_1, t_1 we will henceforth write simply x, z, t.

We seek solutions in the form $e^{ilx}\zeta(z,t)$, i.e. we shall make a Fourier transformation with respect to x. For ζ we obtain

$$\frac{\partial^2}{\partial t^2}\left[\frac{1}{4}\beta^2+l^2-\left(\frac{\partial}{\partial z}-iklt\right)^2\right]\zeta-i\beta kl\frac{\partial\zeta}{\partial t}+g\beta l^2\zeta=0\qquad(1.2)$$

Many authors, including Taylor [5] and the authors of [4], neglect the next to last term in this equation. This term is retained below, although its inclusion does not change the results.

The initial conditions are such that

$$\zeta(z,0)=\varphi(z),\qquad \zeta_t(z,0)=\psi(z)$$

The boundary condition at the surface of the ground is

$$w = 0 \quad \text{or} \quad \zeta(0,\ t) = 0$$

With regard to the conditions at infinity (as $z \to \infty$), such a degree of regularity is required here so as to ensure the existence and uniqueness of the solution. As we shall see, it is sufficient for this purpose to require that ζ grow more slowly at infinity than any exponential function. It is sufficient to solve separately two boundary-value problems, considering $\psi \equiv 0$ and $\phi \equiv 0$. We shall give the solution of the first of these problems; the second one is solved in an analogous manner. We will assume the initial function $\phi(z)$ to be different from zero only in a finite region of space; by virtue of the boundary condition $\phi(0) = 0$.

2. At this point we shall derive a formula for the solution of the Cauchy problem into which a certain, as yet undetermined, function will enter. Following this an integral equation for this function will be derived. In Appendix 1 the equation will be investigated.

We will apply the Laplace transformation

$$\zeta^*(p,\ t) = \int_0^\infty e^{-pz}\,\zeta(z,\ t)\,dz$$

to Equation (1.2).

Multiplying (1.2) by e^{-pz} and integrating with respect to z from 0 to ∞, we obtain

$$\frac{\partial^2}{\partial t^2}[m^2 - (p - iklt)^2]\zeta^* - i\beta kl\,\frac{\partial \zeta^*}{\partial t} + g\beta l^2\zeta^* = \xi(t) \tag{2.1}$$

Here

$$\xi(t) = -\frac{\partial^3 \zeta(0,t)}{\partial z\,\partial t^2}, \qquad m^2 = \frac{1}{4}\beta^2 + l^2$$

We shall designate the Laplace transform of $\phi(z)$ by $\phi^*(z)$. Taking the initial conditions $\zeta^*((p,\ 0) = \phi^*(p)$, $\zeta_t{}^*(p,\ 0) = 0$ into account, we find the solution of Equation (2.1) in the following form:

$$\zeta^*(p,\ t) = \int_0^t \frac{\tau_1{}^*(s - i\varkappa t)\,\tau_2{}^*(s - i\varkappa t_1) - \tau_2{}^*(s - i\varkappa t)\,\tau_1{}^*(s - i\varkappa t_1)}{W\,[1 - (s - i\varkappa t_1)]^{-1+\beta_1}\,[1 + (s - i\varkappa t_1)]^{-1-\beta_1}}\,\xi(t_1)\,dt_1 +$$

$$+ i\varkappa\,\frac{\tau_1{}^*(s - i\varkappa t)\,\tau_2{}^{*\prime}(s) - \tau_2{}^*(s - i\varkappa t)\,\tau_1{}^{*\prime}(s)}{W\,(1 - s)^{-2+\beta_1}\,(1 + s)^{-2-\beta_1}}\,\varphi^*(ms) \tag{2.2}$$

Here $s = p/m$, $\kappa = kl/m$, $\beta_1 = \beta/2m$, and W is a certain constant. The functions $\tau_1{}^*$ and $\tau_2{}^*$ are expressed by the hypergeometric functions

$$\tau_1^*(s) = F\left(\tfrac{3}{2} + r, \tfrac{3}{2} - r; 2 - \beta_1; \tfrac{1-s}{2}\right)$$

$$\left(r = \sqrt{\tfrac{1}{4} - R},\ R = \tfrac{g\beta}{k^2}\right)$$

$$\tau_2^*(s) = (1-s)^{-1+\beta_1} F\left(\tfrac{1}{2} + \beta_1 + r,\ \tfrac{1}{2} + \beta_1 - r;\ \beta_1; \tfrac{1-s}{2}\right)$$

Here R is the Richardson number. It is obvious that for $R < 1/4$ the parameter r has real values and $0 < r \leqslant 1/2$ and that for $R > 1/4$ it has purely imaginary values.

The function $\zeta^*(p, t)$, defined by Formula (2.2), actually represents the Laplace transform of a certain function $\zeta(z, t)$; in addition $\zeta(0, t) = 0$. This follows from the asymptotic expression for $\zeta^*(p, t)$ in the half-plane Re $p > m$ (or Re $s > 1$) where this function is analytic. Indeed, as is easily shown, as $|p| \to \infty$ in this half-plane $\zeta^*(p, t) = 0(|p|^{-2})$. Thus, for all $\xi(t)$ the function $\zeta(z, t)$ satisfies Equation (1.2), the initial conditions and the boundary condition at $z = 0$.

3. At present, the question of determining the function $\xi(t)$ which occurs in Formula (2.2) is left open. We have not yet made use of the boundary condition at $z = \infty$. A direct way of taking account of it would be as follows. After applying the inversion formula to (2.2) and finding $\zeta(z, t)$, one should investigate the asymptote of this solution as $z \to \infty$ and require that the condition of regularity be fulfilled. But such a way entails considerable difficulties. We shall treat the matter differently. Regularity of the function in the half-plane Re $p > 0$ is the necessary condition for the function $\zeta(z, t)$ to grow more slowly than any exponential function. On the other hand, the functions entering into the right-hand side of (2.2) have branch points in this half-plane. It turns out that by choosing $\xi(t)$ it is possible to remove the multivaluedness of this right-hand side (moreover this requirement determines $\xi(t)$ in a unique manner). The function $\zeta^*(p, t)$ becomes regular in the half-plane Re $p > 0$ in this way.

The hypergeometric function has a unique branch point when the value of its argument is equal to unity. Thus, $\tau_1^*(s)$ is regular in the half-plane Re $s > 0$, and τ_2^* is the product of $(1 - s)^{-1+\beta_1}$ by a function which is regular for Re $s > 0$. Therefore, the multivaluedness of the right-hand side of (2.2) appears in the calculation of the terms

$$\tau_2^*(s - i\varkappa t) \int_0^t \frac{\tau_1^*(s - i\varkappa t_1)\, \xi(t_1)\, dt_1}{W[1 - (s - i\varkappa t_1)]^{-1+\beta_1} [1 + (s - i\varkappa t_1)]^{-1-\beta_1}} +$$

$$+ \tau_2^*(s - i\varkappa t) \frac{i\varkappa \tau_1^{*\prime}(s)\, \varphi^*(ms)}{W(1-s)^{-2-\beta_1}(1+s)^{-2+\beta_1}}$$

This expression, generally speaking, varies with the path about a closed contour in the half-plane Re $s > 0$ which contains within it points of the segment $[1, 1 + i\kappa t]$.

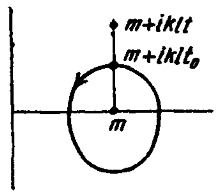

FIG. 1.

We shall find the increase of this expression with a path about the contour shown in Fig. 1. The circuit begins at some point $s = 1 + i\kappa t_0$ $(0 < t_0 < t)$ and ends at the same point. All multivalued functions $[1 - (s - i\kappa t_1)]^{-\beta_1}$, where $0 < t_1 < t_0$, increase with such a path by one and the same amount. We shall equate the increase of the expression written above to zero:

$$\int_0^{t_0} \frac{F\,(^3/_2 + r,\ ^3/_2 - r;\ 2 - \beta_1;\ -i\varkappa\,(t_0 - t_1)\,/\,2)\,\xi\,(t_1)\,dt_1}{[-\,i\varkappa\,(t_0 - t_1)]^{-1+\beta_1}\,[2 + i\varkappa\,(t_0 - t_1)]^{-1-\beta_1}} -$$
$$- \frac{i\varkappa}{2}\,\frac{F'(^3/_2 + r,\ ^3/_2 - r;\ 2 - \beta_1;\ -i\varkappa t_0\,/\,2)\,\varphi^*\,(m + iklt_0)}{(-\,i\varkappa t_0)^{-2+\beta_1}\,(2 + i\varkappa t_0)^{-2-\beta_1}} = 0$$

or, making use of the formula

$$F\,(a,\ b;\ c;\ z) = (1 - z)^{c-a-b} F\,(c - a,\ c - b;\ c;\ z)$$

we will have

$$\int_0 (t - t_1)^{1-\beta_1} F\left(\tfrac{1}{2} - \beta_1 + r,\ \tfrac{1}{2} - \beta_1 - r;\ 2 - \beta_1;\ -\tfrac{i\varkappa}{2}\,(t - t_1)\right) \xi\,(t_1)\,dt_1 = g\,(t) \quad (3.1)$$

where

$$g\,(t) = Ct^{2-\beta_1}\,(2 + i\varkappa t)^{2+\beta_1} F'\left(\tfrac{3}{2} + r;\ \tfrac{3}{2} - r;\ 2 - \beta_1;\ -\tfrac{i\varkappa t}{2}\right) \varphi^*\,(m + iklt)$$

For determining $\xi(t)$ we have obtained an integral equation of the Volterra type of the second kind with a difference kernel.

4. In order to obtain a formula for the solution to our problem, it remains to carry out the inverse Laplace transformation of Equation (2.2), i.e. to pass to the inverse form. It is not necessary to express these inverse forms in terms of well-known special functions. For our purpose it is sufficient to know that in each individual case such an inverse function actually exists. The latter is easily verified by a well-known theorem on the representability of functions by a Laplace integral.

We shall integrate (2.2) by parts; we have

$$\zeta^*(p, t) = \tau_1^*(s - i\varkappa t)\int_0^t \chi_2^*(s - i\varkappa t_1)\int_0^{t_1} \xi(t_2)\,dt_2 dt_1 - \tag{4.1}$$
$$- \tau_2^*(s - i\varkappa t_1)\int_0^t \chi_2^*(s - i\varkappa t_1)\int_0^{t_1} \xi(t_2)\,dt_2 dt_1 + \tau_1^*(s - i\varkappa t)\,\eta_2^*(s) - \tau_2^*(s - i\varkappa t)\eta_1^*(s)$$

Here

$$\chi_{1,2}^*(s) = \frac{i\varkappa\tau_{1,2}^*(s)}{W(1-s)^{-1+\beta_1}(1+s)^{-1-\beta_1}}, \qquad \eta_{1,2}^*(s) = \frac{i\varkappa\tau_{1,2}^{*\prime}(s)\,\varphi^*(s)}{W(1-s)^{-2+\beta_1}(1+s)^{-2-\beta_1}}$$

All functions of s entering in the right-hand side of (4.1), i.e. $\tau_{1,2}^*$, $\chi_{1,2}^*$, $\eta_{1,2}^*$, are Laplace transforms of certain functions $\tau_{1,2}(z)$, $\chi_{1,2}(z)$, $\eta_{1,2}(z)$, which follows from their asymptotic behavior for large values of $|s|$ where $\mathrm{Re}\, s > 0$. Namely

$$|\tau_{1,2}^*(s)| < K|s|^{-3/2+\mathrm{Re}\, r}, \quad |\chi_{1,2}^*(s)| < K|s|^{-1/2+\mathrm{Re}\, r}, \quad |\eta_{1,2}^*(s)| < K|s|^{-1/2+\mathrm{Re}\, r}$$

while their derivatives tend to zero by an order more quickly.

Hence, in particular

$$|\tau_{1,2}(z)| < K|z|^{1/2-\mathrm{Re}\, r}, \qquad |\chi_{1,2}(z)| < K|z|^{-1/2-\mathrm{Re}\, r} \quad \text{for small value of } |z|$$

Using the convolution theorem, we will have

$$\zeta(z, t) = e^{i\varkappa tz}\int_0^z [\tau_1(z - z_1)\chi_2(z) - \tau_2(z - z_1)\chi_1(z_1)]\,\vartheta_{z_1}(t)\,dz_1 +$$
$$+ \int_0^z e^{i\varkappa t(z-z_1)}[\tau_1(z - z_1)\eta_2(z_1) - \tau_2(z - z_1)\eta_1(z_1)]\,dz_1 \tag{4.2}$$

Here

$$\vartheta_z(t) = \int_0^t e^{-i\varkappa(t-t_1)z}\int_0^{t_1} \xi(t_2)\,dt_2\,dt_1$$

Now we can see that the stability of the flow, i.e. the behavior of $\zeta(z, t)$ as $t \to \infty$, depends only on the characteristics of $\vartheta_z(t)$ or $\xi(t)$ as the argument increases without limit. In Appendix 1, where the integral equation (3.1) is investigated, it is demonstrated that

$$|\vartheta_z(t)| < K \qquad \text{for } r \text{ real}$$
$$|\vartheta_z(t)| < K_1 + K_2 |\lg(K_3/t + z)| \qquad \text{for } r \text{ imaginary} \qquad (4.3)$$

From this it is easy to be satisfied about the boundedness of $\zeta(z, t)$ as $t \to \infty$; for r real the integral in (4.2) is evaluated as

$$K \int_0^z z_1^{-1/2-r} (z - z_1)^{-1/2-r} dz_1$$

where K does not depend on t. For r imaginary the estimate is

$$K \int_0^z z_1^{-1/2} |\lg(K_3/t + z_1)| dz_1$$

The latter expression is bounded as $t \to \infty$. Thus, the stability theorem has been proved.

5. *Appendix 1*. We shall make an investigation of the integral equation (3.1) omitting some details of the cumbersome calculations.

For any $\xi(t)$ the left-hand side of (3.1) together with its derivative vanishes for $t = 0$. Consequently, it is necessary for the existence of the solution that the right-hand side $g(t)$ also have the same characteristic. But in our case this condition is satisfied. Soon it will be clear that this is also sufficient for the existence of the solution. We will solve the equation by the Laplace transform method. To do this we will multiply both sides of the equation by $e^{-\sigma t}$ and integrate with respect to t from 0 to ∞. As will be shown in Appendix 2, the Laplace transform of the function $t^{1-\beta_1} F(1/2 - \beta_1 + r, 1/2 - \beta_1 - r; 2 - \beta_1; -1/2 i\varkappa t)$ is

$$K\sigma^{-2} e^{-i\sigma/\varkappa} W_{\beta_1, r}(2e^{-i\pi/2}\sigma/\varkappa)$$

where $W_{\beta r}$ is the Whittaker function for $|\arg \sigma| < 1/2\, \pi$.

Hence, it is not difficult to find the Laplace transform of $\xi(t)$, which we shall designate as $\xi^*(\sigma)$:

$$\xi^*(\sigma) = \frac{\sigma^2 g^*(\sigma)}{K e^{-i\sigma/\varkappa} W_{\beta_1, r}(2e^{-i\pi/2}\sigma/\varkappa)} \qquad (5.1)$$

Using the asymptote of the Whittaker function for large values of $|\sigma|$ and the fact that $g^*(\sigma) = 0(|\sigma|^{-3})$ (since $g(0) = g'(0) = 0$), we find that $\xi^*(\sigma) = 0(|\sigma|^{-\beta_1 - 1})$ as $|\sigma| \to \infty$ for $|\arg \sigma| < 1/2\, \pi$. It will be further shown that $\xi^*(\sigma)$ is regular in the right-hand half-plane. Consequently, $\xi^*(\sigma)$ which is defined by Formula (4.3) is actually the Laplace transform of a certain function $\xi(t)$ which satisfies the integral equation.

We will pass to the evaluation of $\xi(t)$ as $t \to \infty$. It is not really the function $\xi(t)$ itself which is of interest to us, but rather the function

$$\vartheta_z(t) = \int_0^t e^{-i\varkappa(t-t_1)z} \int_0^{t_1} \xi(t_2)\, dt_2 dt_1$$

Its Laplace transform is

$$\vartheta_z^*(\sigma) = \frac{\sigma g^*(\sigma)}{K(\sigma + i\varkappa z)\, e^{-i\sigma/\varkappa} W_{\beta_1, r}(2e^{-i\pi/2}\sigma/\varkappa)}$$

Therefore

$$\vartheta_z(t) = \int_0^t g(t-t_1) R_z(t_1)\, dt_1$$

where

$$R_z^*(\sigma) = \frac{\sigma}{K(\sigma + i\varkappa z)\, e^{-i\sigma/\varkappa} W_{\beta_1, r}(2e^{-i\pi/2}\sigma/\varkappa)}$$

This function has a pole at the point $\sigma = -i\kappa z$. We shall designate its residue by $b(z)$. It is not difficult to obtain an estimate of the value of this residue for small values of z by using the asymptote of the Whittaker function in the neighborhood of zero, namely

$$|b(z)| < Kz^{1/2 + \mathrm{Re}\, r}$$

It is more complicated to ascertain the location of the poles which arise from the zeros of the Whittaker function. In the case of r real the function

$$W_{\beta_1, r}(2e^{-i\pi/2}\sigma/\varkappa) \quad \text{in the sector} \quad -\tfrac{1}{2}\pi \leqslant \arg\sigma \leqslant \pi$$

does not have any zeros with the possible exception of one on the radius $\arg\sigma = 1/2\,\pi$.

In the case of r imaginary there is a denumerable set of zeros $\sigma^{(n)}$ on the radius $\arg\sigma = 1/2\,\pi$ which accumulate at the origin of the coordinate system; here

$$\sigma^{(n)} \sim Ke^{i\pi/2} e^{-\pi n/|r|}.$$

There are no other zeros in the sector $-1/2\,\pi < \arg\sigma < \pi$. We shall designate the residues at $\sigma = \sigma^{(n)}$ by $a_n(z)$. It is possible to give an estimate

$$|a_n(z)| < K|\sigma^{(n)}|^{1/2} / (|\sigma^{(n)}| + \varkappa z) \quad (K \text{ does not depend on } z)$$

In the sector $-\pi + \epsilon < \arg z < -\pi/2 (\epsilon > 0)$ a finite number of zeros can no longer be found. Still we note that as $|\sigma| \to \infty$ the estimate

$$R_z^*(\sigma) = O(|\sigma|^{-\beta_1}) \qquad \text{in the sector} \quad |\arg \sigma| < \pi - \varepsilon$$

is correct.

According to the inversion formula of the Laplace transform we have

$$R_z(t) = \frac{1}{2\pi i} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} \frac{\sigma e^{\sigma t} d\sigma}{K(\sigma + i\varkappa z)\, e^{-i\sigma/\varkappa} W_{\beta_1, r}(2e^{-i\pi/2}\sigma / \varkappa)}$$

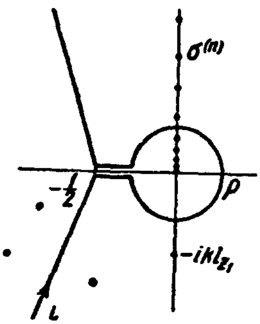

FIG. 2.

We shall now deform the contour as shown in Fig.2. The slope of the straight lines is arbitrary. We choose the radius of the circle in dependence on z and t so that $c_1/t < \rho < c_1/t$, $|\rho - \kappa z| > c_3\rho$, where c_1, c_2, c_3 are fixed constants. If r is an imaginary number, we add one further requirement – the circle must intersect the imaginary axis exactly midway between the roots of the Whittaker function which are distributed along this axis.

With this choice of ρ the denominator of the integrand exceeds $K\rho^{3/2 - \mathrm{Re}\, r}$ along the modulus of the circle. One can satisfy oneself on this point using the asymptote of the Whittaker function. In this case the integrand is less than $K\rho^{-1/2 + \mathrm{Re}\, r}$, and the integral is less than $K\rho^{1/2 + \mathrm{Re}\, r}$, i.e. less than $Kt^{-1/2 - \mathrm{Re}\, r}$ and, consequently, it vanishes as $t \to \infty$. Exactly the same estimate is obtained for the integrals along the horizontal segments. The integral along the sloping lines vanishes exponentially. With this deformation of the contour the residues are isolated. We have

$$R_z(t) = b(z)\, e^{-i\varkappa z t} + \sum_{|\sigma^{(n)}| > \rho} a_n(z)\, e^{i|\sigma^{(n)}| t} + O(t^{-1/2 - \mathrm{Re}\, r})$$

Residues from poles in the region $-\pi + \epsilon < \arg \sigma < -1/2\,\pi$ enter also into the remainder term, if there are such poles, since their residues vanish exponentially as $t \to \infty$. For r real, instead of the sum which appears in the second position, there should be only one term. The necessary estimate (4.3) for $\vartheta_z(t)$ is now obtained without any difficulty from

$$\vartheta_z(t) = \int\limits_0^t g(t - t_1)\, R_z(t_1)\, dt_1, \qquad g(t) = O(t^{-1/2 + \mathrm{Re}\, r})$$

and from the fact that $\sigma^{(n)}$ vanishes exponentially as n increases, and also from the estimate of the residues $a_n(z)$ and $b(z)$.

Note. If we were to make the same simplifications as in [5,6], which were mentioned at the beginning, i.e. if we were to neglect the next to last term in Formula (1.2), the results would not be changed except that

$$W_{0,r}(\sigma) = \sqrt{\frac{2}{\pi}}\, K_r\left(\frac{\sigma}{2}\right)$$

would enter everywhere in place of $W_{\beta_1, r}$.

The density stratification introduces two dimensionless parameters, which correspond to the inertial and Archimedian effects of this stratification. The indicated simplification is the neglect of the inertial effect.

6. *Appendix 2.* We shall prove that the Laplace transform of the hypergeometric function

$$t^{d-1-\beta}\, F\left(\tfrac{1}{2} - \beta + r, \tfrac{1}{2} - \beta - r; d - \beta; -\alpha t\right) \qquad (\mathrm{Re}\,(d - \beta) > 0)$$

is

$$\Gamma(d-\beta)\, \alpha^{\beta}\sigma^{-d} e^{\sigma/2\alpha} W_{\beta, r}(\sigma/\alpha), \qquad (|\arg \alpha| < \pi, \ |\arg \sigma/\alpha| < \pi)$$

For the demonstration we shall use the Barnes-Mellin integral [8,p.71]

$$f(t) \equiv t^{d-1-\beta} F\left(\tfrac{1}{2} - \beta + r, \tfrac{1}{2} - \beta - r; d - \beta; -\alpha t\right) = \frac{\Gamma(d-\beta)}{\Gamma(\frac{1}{2} - \beta - r)\,\Gamma(\frac{1}{2} - \beta + r)} \times$$

$$\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2} - \beta - r + s)\,\Gamma(\frac{1}{2} - \beta - r + s)\,\Gamma(-s)}{\Gamma(d - \beta + s)} (\alpha t)^s t^{d-1-\beta}\, ds$$

The integration path is such that the poles of the function $\Gamma(\frac{1}{2} - \beta - r + s) \times \Gamma(\frac{1}{2} - \beta + r + s)$, i.e. the points $s = \beta - \frac{1}{2} \pm r - n$ $(n = 0, 1, 2, \ldots)$, lie to the left of the path, and the poles of $\Gamma(-s)$, i.e. the points $s = 0, 1, 2, \ldots,$ lie to the right of the integration path. We shall apply the Laplace transformation to both sides of this equality

$$f^*(\sigma) = \int_0^{\infty} e^{-\sigma t} f(t)\, dt =$$

$$= \Gamma(d-\beta) \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2} - \beta - r + s)\,\Gamma(\frac{1}{2} - \beta + r + s)\,\Gamma(-s)}{\Gamma(\frac{1}{2} - \beta - r)\,\Gamma(\frac{1}{2} - \beta + r)} \alpha^s \sigma^{-s-d+\beta}\, ds$$

We shall make the variable substitution $s = -s_1$:

$$f^*(\sigma) = \Gamma(d-\beta)\, \sigma^{\beta - d} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{1}{2} - \beta - r - s)\,\Gamma(\frac{1}{2} - \beta + r - s)\,\Gamma(s)}{\Gamma(\frac{1}{2} - \beta - r)\,\Gamma(\frac{1}{2} - \beta + r)} (\sigma/\alpha)^s\, ds$$

There remains only to recall the Barnes-Mellin formula for the

Whittaker function [8, p. 148]

$$W_{\beta,r}(\sigma/\alpha) = \frac{e^{-\sigma/2\alpha}(\sigma/\alpha)^{\beta}}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(1/2-\beta-r-s)\,\Gamma(1/2-\beta+r-s)\,\Gamma(s)}{\Gamma(1/2-\beta-r)\,\Gamma(1/2-\beta+r)}(\sigma/\alpha)^{s}\,ds$$

which proves our assertion.

BIBLIOGRAPHY

1. Keldysh, M.V., O sobstvennykh znacheniiakh i sobstvennykh funktsiiakh nekotorykh klassov nesamosopriazhennykh uravnenii (On the eigen-values and eigen-functions of certain classes of non-self-conjugate equations). *Dokl. Akad. Nauk* Vol. 77, No. 1, 1951.
2. Lin, C.C., *Teoriia gidrodinamicheskoi ustoichivosti (Theory of Hydro-dynamic Stability)*. Izd-vo inostr. lit-ry, 1958.
3. Schlichting, H., Turbulenz bei Warmeschichtung. *ZAMM* Vol. 15, p.313, 1935.
4. Eliassen, A., Hoiland, E. and Riis, E., *Two-dimensional Perturbation of a Flow with Constant Shear*. Inst. for weather and climate res., Publ. 1, Oslo, 1953.
5. Taylor, G.I., Effect of variation in density on the stability. *Proc. Roy. Soc.* A, Vol. 132, p. 820, 1931.
6. Schlichting, H., *Teoriia pogranichnogo sloia (Boundary-Layer Theory)*. Izd-vo inostr. lit-ry, 1956.
7. Prandtl, *Gidroaeromekhanika (Hydro- and Aeromechanics)*.
8. Whittaker, E.T. and Watson, G.N., *Kurs sovremennogo analiza (Course in Modern Analysis)*. Vol. 2, GTTI, 1934.

Translated by R.D.C.